

# ADDITIVITY OF THE IDEAL OF MICROSCOPIC SETS

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**ABSTRACT.** A set  $M \subset \mathbb{R}$  is microscopic if for each  $\varepsilon > 0$  there is a sequence of intervals  $(J_n)_{n \in \omega}$  covering  $M$  and such that  $|J_n| \leq \varepsilon^{n+1}$  for each  $n \in \omega$ . We show that there is a microscopic set which cannot be covered by a sequence  $(J_n)_{n \in \omega}$  with  $\{n \in \omega : J_n \neq \emptyset\}$  of lower asymptotic density zero. We prove (in ZFC) that additivity of the ideal of microscopic sets is  $\omega_1$ . This solves a problem of G. Horbaczewska. Finally, we discuss additivity of some generalizations of this ideal.

## 1. INTRODUCTION

For  $n \in \omega$  we use the identification  $n = \{0, 1, \dots, n-1\}$ . By  $\text{card}(A)$  we denote cardinality of a set  $A$ . For an interval  $I \subset \mathbb{R}$  by  $|I|$  we denote its length. Given  $r \in \mathbb{R}$  and  $A \subset \mathbb{R}$  we write  $r \cdot A = \{ra : a \in A\}$  and  $r + A = \{r + a : a \in A\}$ .

We say that a sequence of intervals  $(J_n)_{n \in \omega}$  *covers the set*  $A \subset \mathbb{R}$  if  $A \subset \bigcup_{n \in \omega} J_n$ .

**Definition 1.1** (J. Appell, [1]). A set  $M \subset \mathbb{R}$  is called *microscopic* if for each  $\varepsilon > 0$  there exists a sequence of intervals  $(J_n)_{n \in \omega}$  covering  $M$  and such that  $|J_n| \leq \varepsilon^{n+1}$  for each  $n \in \omega$ . The family of all microscopic sets will be denoted by  $\mathcal{M}$ .

This notion was introduced in 2001 by J. Appell in [1]. Deeper studies of microscopic sets were done by J. Appell, E. D’Aniello and M. V  th in [2]. Since that time, several papers were devoted to this subject, including [9] [10] and [11]. In [8] one can find a summary of the progress made in this area.

It is easy to see that every microscopic set is contained in some microscopic  $\mathbf{G}_\delta$ -set, i.e.,  $\mathcal{M}$  is  $\mathbf{G}_\delta$ -generated (cf. [8, Theorem 1.1]). Moreover,  $\mathcal{M}$  is strictly smaller than the  $\sigma$ -ideal of sets of Lebesgue measure zero (cf. [8]). Therefore, many classical theorems stating that some property holds everywhere except a set of Lebesgue measure zero, are being strengthened by showing that actually the set of exceptions can be chosen to be microscopic. For instance, it can be proved that  $\mathbb{R}$  can be decomposed into two sets such

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that one of them is of first category and the second one is microscopic (cf. [9]).

The aim of this paper is to determine the smallest number of sets from  $\mathcal{M}$  union of which is not in  $\mathcal{M}$  anymore. For this purpose, we need the notion of asymptotic density of a subset of  $\omega$ .

Recall that for any  $A \subset \omega$  its *upper and lower asymptotic density* are given by the formulas:

$$\begin{aligned}\overline{d}(A) &= \limsup_{j \rightarrow \infty} \frac{\text{card}(A \cap (j+1))}{j+1}, \\ \underline{d}(A) &= \liminf_{j \rightarrow \infty} \frac{\text{card}(A \cap (j+1))}{j+1}.\end{aligned}$$

If  $\overline{d}(A) = \underline{d}(A)$ , then we say that the set  $A$  is of asymptotic density  $d(A)$  which is equal to this common value.

**Definition 1.2.** Let  $\delta \in [0, 1]$ . We say that a microscopic set  $M \subset \mathbb{R}$  admits a cover of (lower) asymptotic density  $\delta$  if for every  $\varepsilon > 0$  there is  $D \subset \omega$  with  $d(D) \leq \delta$  ( $\underline{d}(D) \leq \delta$ ) and a sequence of intervals  $(J_d)_{d \in D}$  which covers  $M$  and satisfies  $|J_d| \leq \varepsilon^{d+1}$  for all  $d \in D$ .

**Remark 1.3.** It is easy to see that any microscopic set  $M \subset \mathbb{R}$  admits a cover of arbitrarily small positive asymptotic density. Actually, for any  $k \in \omega$  and  $\varepsilon > 0$  one can find a sequence of intervals  $(J_d)_{d \in D}$ , where  $D = (k+1) \cdot (\omega+1)$ , which covers  $M$  and satisfies  $|J_d| \leq \varepsilon^{d+1}$  for each  $d \in D$ .

Indeed, set any  $k \in \omega$  and  $\varepsilon > 0$ . Since  $M$  is microscopic, there is a sequence of intervals  $(J'_n)_{n \in \omega}$  covering  $M$  with  $|J'_n| \leq (\varepsilon^{k+1})^{n+1} = \varepsilon^{(k+1)(n+1)}$  for each  $n \in \omega$ . Then it suffices to put  $J_{(k+1)(n+1)} = J'_n$  for  $n \in \omega$ .

In Section 3 we will show that the above cannot be strengthened, i.e., there is a microscopic set which does not admit a cover of lower asymptotic density zero (cf. Theorem 3.1).

From Remark 1.3 it easily follows that  $\mathcal{M}$  is a  $\sigma$ -ideal (see [2] or [8] for details). Among studies of  $\sigma$ -ideals, examination of cardinal coefficients related to them has been of great interest during last decades. This is due to the famous Cichoń's diagram which classifies cardinal coefficients of the ideals of null sets and meager sets (cf. [3] and [6]).

Recall the definitions of *additivity*, *covering number*, *uniformity number* and *cofinality* of an ideal  $\mathcal{I}$  of subsets of  $\mathbb{R}$ :

$$\begin{aligned}\text{add}(\mathcal{I}) &= \min \left\{ \text{card}(\mathcal{A}) : \mathcal{A} \subset \mathcal{I} \quad \wedge \quad \bigcup \mathcal{A} \notin \mathcal{I} \right\}; \\ \text{cov}(\mathcal{I}) &= \min \left\{ \text{card}(\mathcal{A}) : \mathcal{A} \subset \mathcal{I} \quad \wedge \quad \bigcup \mathcal{A} = \mathbb{R} \right\};\end{aligned}$$

$$\begin{aligned}\mathbf{non}(\mathcal{I}) &= \min \{ \text{card}(A) : A \subset \mathbb{R} \ \wedge \ A \notin \mathcal{I} \}; \\ \mathbf{cof}(\mathcal{I}) &= \min \{ \text{card}(\mathcal{B}) : \mathcal{B} \subset \mathcal{I} \ \wedge \ \forall A \in \mathcal{I} \exists B \in \mathcal{B} A \subset B \}.\end{aligned}$$

One can easily prove the following inequalities:

$$\mathbf{add}(\mathcal{I}) \leq \mathbf{non}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I}) \quad \text{and} \quad \mathbf{add}(\mathcal{I}) \leq \mathbf{cov}(\mathcal{I}) \leq \mathbf{cof}(\mathcal{I}).$$

For more on cardinal coefficients see e.g. [3] or [6].

For the ideal of microscopic sets each of those cardinal coefficients lies between  $\omega_1$  and  $2^\omega$  (possibly is equal to one of those two numbers), since  $\mathcal{M}$  is a  $\sigma$ -ideal of subsets of  $\mathbb{R}$  containing all singletons and  $\mathbf{G}_\delta$ -generated. The aim of this paper is to determine additivity of the ideal of microscopic sets. This problem was posed in 2010 by G. Horbaczewska in her talk *Properties of the  $\sigma$ -ideal of microscopic sets* during XXIV Summer Conference on Real Functions Theory in Stara Lesna, Slovakia.

Firstly, let us discuss the last three coefficients in the case of microscopic sets. By  $\mathcal{N}$  we denote the family of sets of Lebesgue measure zero. Recall that a set  $S \subset \mathbb{R}$  is *of strong measure zero* if for each sequence of positive reals  $(\varepsilon_n)_{n \in \omega}$  there exists a sequence of intervals  $(J_n)_{n \in \omega}$  covering  $S$  and such that  $|J_n| \leq \varepsilon_n$  for each  $n \in \omega$ . The family of sets of strong measure zero will be denoted by  $\mathcal{S}$ .

It is well known that both  $\mathcal{N}$  and  $\mathcal{S}$  are  $\sigma$ -ideals. One can easily see that  $\mathcal{S} \subset \mathcal{M} \subset \mathcal{N}$ . In fact, both of these inclusions are proper (cf. [8]).

**Remark 1.4.** Assume Martin's axiom (cf. [12]). Then  $2^\omega = \mathbf{non}(\mathcal{M}) = \mathbf{cov}(\mathcal{M}) = \mathbf{cof}(\mathcal{M})$ . Indeed, under Martin's axiom  $2^\omega = \mathbf{add}(\mathcal{N}) = \mathbf{add}(\mathcal{S})$  (cf. [4, Theorem 2.1] and [12, Theorem 2.21]). Since  $\mathcal{S} \subset \mathcal{M} \subset \mathcal{N}$ , we also have  $\mathbf{cov}(\mathcal{N}) \leq \mathbf{cov}(\mathcal{M})$  and  $\mathbf{non}(\mathcal{S}) \leq \mathbf{non}(\mathcal{M})$ . Hence,

$$2^\omega = \mathbf{add}(\mathcal{N}) \leq \mathbf{cov}(\mathcal{N}) \leq \mathbf{cov}(\mathcal{M}) \leq \mathbf{cof}(\mathcal{M}) \leq 2^\omega$$

and

$$2^\omega = \mathbf{add}(\mathcal{S}) \leq \mathbf{non}(\mathcal{S}) \leq \mathbf{non}(\mathcal{M}) \leq 2^\omega.$$

Although  $\mathbf{non}(\mathcal{M})$ ,  $\mathbf{cov}(\mathcal{M})$  and  $\mathbf{cof}(\mathcal{M})$  may all be equal to  $2^\omega$ , we will prove in Section 3 that  $\mathbf{add}(\mathcal{M})$  is always equal to  $\omega_1$  (cf. Theorem 3.2).

The paper is organized as follows. In Section 2 we deal with a technical construction which will be helpful in further considerations. In Section 3 we use methods developed in Section 2 to construct a microscopic set which does not admit a cover of lower asymptotic density zero and to prove (in ZFC) that additivity of the ideal of microscopic sets is  $\omega_1$ . Section 4 is devoted to some generalizations of the ideal of microscopic sets and their additivity.

## 2. SPACING ALGORITHM

**Definition 2.1.** Given two sequences of intervals  $(I_a)_{a \in A}$  and  $(J_d)_{d \in D}$  the set  $Y((I_a)_{a \in A}, (J_d)_{d \in D})$  consists of all  $a \in A$  with the following property:

$$\forall d \in D \left( I_a \cap J_d \neq \emptyset \Rightarrow \forall_{\substack{a' \in A \\ a \neq a'}} I_{a'} \cap J_d = \emptyset \right).$$

In the proofs of Theorems 3.1 and 3.2 the following technical lemma will be crucial.

**Lemma 2.2** (Spacing Algorithm). *Let  $I$  be an interval of length  $\frac{1}{7^m}$  for some  $m \in \omega$ . Suppose that  $A \subset \omega$  is of positive density and  $\min A > m$ . Then one can define a sequence of intervals  $(I_a)_{a \in A}$  with  $|I_a| = \frac{1}{7^a}$  and  $I_a \subset I$  for all  $a \in A$ , in such a way that given any  $D \subset \omega \setminus m$  and a sequence of intervals  $(J_d)_{d \in D}$ , with  $|J_d| \leq \frac{1}{7^{d+1}}$  for all  $d \in D$ , for any  $s \in \omega$  and  $r_0, \dots, r_s \in (0, 1) \setminus \mathbb{Q}$  the set  $Z$  consisting of those  $a \in Y((I_a)_{a \in A}, (J_d)_{d \in D})$  which additionally satisfy*

$$\forall d \in D (I_a \cap J_d \neq \emptyset \Rightarrow \forall_{i \leq s} \forall_{a' \in A} (r_i + I_{a'}) \cap J_d = \emptyset),$$

*is of lower asymptotic density at least  $\frac{d(A)}{4}$ .*

*Proof.* The proof is divided into five parts. At first, we deal with the construction of the intervals  $I_a$  for  $a \in A$ . Then we focus on preliminary discussion concerning calculation of  $\underline{d}(Z)$ . The last three parts are devoted to some technical aspects of this calculation.

**Construction of the intervals  $I_a$  for  $a \in A$ .**

Let  $\varepsilon = \frac{1}{7}$ . Firstly, we construct auxiliary intervals  $K_j^i$  for  $i \in \omega$  and  $j < 4 \cdot 3^i$ . Let  $K_j^0$  for  $j < 4$  be such that:

- each of them is of length  $\varepsilon^{m+1}$ ;
- the distance between each two of them is at least  $\varepsilon^{m+1}$ ;
- each of them is contained in  $I$ ;
- $\inf K_0^0 = \inf I$  and  $\sup K_1^0 = \sup I$ .

Suppose that  $K_j^i$  for all  $i < k$  and  $j < 4 \cdot 3^i$  are defined. Let  $K_j^k$  for  $j < 4 \cdot 3^k$  be such that:

- each of them is of length  $\varepsilon^{k+m+1}$ ;
- the distance between each two of them is at least  $\varepsilon^{k+m+1}$ ;
- $K_j^k$  is contained in  $K_l^{k-1}$ , where  $l = j \bmod 3 \cdot 3^{k-1}$ ;
- $\inf K_l^k = \inf K_l^{k-1}$  and  $\sup K_{3^k+l}^k = \sup K_l^{k-1}$ .

Now we can proceed to the construction of the intervals  $I_a$  for  $a \in A$ . Let  $\{a_0, a_1, \dots\}$  be an increasing enumeration of the set  $A$ . Define also the

family of intervals

$$\mathcal{K} = \{K_j^i : i \in \omega \text{ and } 3 \cdot 3^i \leq j < 4 \cdot 3^i\}.$$

Note that for each  $K_j^i$  belonging to  $\mathcal{K}$  there are no  $i' > i$  and  $l < 4 \cdot 3^i$  with  $K_l^{i'}$  contained in  $K_j^i$ . Let  $\{K_0, K_1, \dots\}$  be an enumeration of  $\mathcal{K}$  with  $|K_i| \geq |K_{i+1}|$ . For each  $i$  pick  $I_{a_i}$  to be any interval of length  $\varepsilon^{a_i}$  contained in  $K_i$  ( $|K_i| \geq \varepsilon^{m+i+1} \geq \varepsilon^{a_i}$  since  $\min A > m$ ).

Observe that for any  $i \in \omega$  and  $j < 3 \cdot 3^i$  density of the set  $\{a \in A : I_a \subset K_j^i\}$  is equal to  $d(A)/(3 \cdot 3^i)$ .

**Calculation of  $\underline{d}(Z)$ .**

We are ready to prove that the intervals  $I_a$  for  $a \in A$  are as needed. Consider any  $s \in \omega$  and  $r_0, \dots, r_s \in (0, 1) \setminus \mathbb{Q}$ . Set also  $D \subset \omega \setminus m$  and a sequence of intervals  $(J_d)_{d \in D}$  with  $|J_d| \leq \frac{1}{7^{d+1}}$  for all  $d \in D$ .

Let  $t_n = (3^0 + 3^1 + \dots + 3^n) - 1 = \frac{3^{n+1} - 3}{2}$  and  $L_n = \{a_{t_n+1}, a_{t_n+1}, \dots, a_{t_{n+1}}\}$  for each  $n \in \omega$ . The sets  $L_n$  are picked in such a way that given  $a \in L_n$  the interval  $I_a$  is contained in  $K_j^{n+1}$  for some  $3 \cdot 3^{n+1} \leq j < 4 \cdot 3^{n+1}$ .

We will show that for any  $\delta > 0$  we have

$$(2.1) \quad \frac{\text{card}(Z \cap (a_{t_{n+1}} + 1) \setminus (a_{t_n} + 1))}{\text{card}(A \cap (a_{t_{n+1}} + 1) \setminus (a_{t_n} + 1))} > \frac{1}{2} - \delta$$

for sufficiently large  $n$  (equivalently: at least  $\frac{1}{2} - \delta$  of all  $a \in L_n$  are in  $Z$  whenever  $n$  is sufficiently large). Once this is done, we conclude that:

$$(2.2) \quad \liminf_{n \rightarrow \infty} \frac{\text{card}(Z \cap (a_{t_n} + 1))}{\text{card}(A \cap (a_{t_n} + 1))} \geq \frac{1}{2}$$

and hence:

$$\liminf_{n \rightarrow \infty} \frac{\text{card}(Z \cap (a_{t_n} + 1))}{a_{t_n} + 1} \geq \frac{d(A)}{2}.$$

Consider now  $a_{t_n} < j < a_{t_{n+1}}$ . Recall the definition of  $t_n$ 's and observe that  $\lim_{n \rightarrow \infty} \frac{t_{n+1} - t_n}{2} = t_n$ . By (2.2) and the fact that  $\text{card}(A \cap (a_{t_n} + 1)) = t_n$ , we get that:

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{\text{card}(Z \cap (j + 1))}{\text{card}(A \cap (j + 1))} \geq \\ & \liminf_{n \rightarrow \infty} \frac{\text{card}(Z \cap (a_{t_n} + 1))}{\text{card}(A \cap (a_{t_n} + 1)) + \frac{1}{2} \text{card}(A \cap ((a_{t_{n+1}} + 1) \setminus (a_{t_n} + 1)))} \\ & = \liminf_{n \rightarrow \infty} \frac{\text{card}(Z \cap (a_{t_n} + 1))}{2 \cdot \text{card}(A \cap (a_{t_n} + 1))}. \end{aligned}$$

It follows that  $\underline{d}(Z) \geq \frac{d(A)}{4}$ .

Therefore, it suffices to prove (2.1), i.e., that for any  $\delta > 0$  at least  $\frac{1}{2} - \delta$  of all  $a \in L_n$  are in  $Z$  whenever  $n$  is sufficiently large. Denote  $Y = Y((I_a)_{a \in A}, (J_d)_{d \in D})$  (cf. Definition 2.1) and let  $A'$  consist of those  $a \in A$  with  $(r_i + I_{a'}) \cap I_a = \emptyset$  for all  $i \leq s$  and  $a' \in A$ . Set  $\delta > 0$ .

The remaining part of the proof is divided into three steps. At first, we show that at least one half of all  $a \in L_n$  is in  $Y$  (for all  $n \in \omega$ ). Then we prove that for sufficiently large  $n$  at least  $1 - \delta$  of all  $a \in L_n$  is in  $A'$ . These two steps together show that for sufficiently large  $n$  at least  $\frac{1}{2} - \delta$  of all  $a \in L_n$  is in  $Y \cap A'$ . Finally, in the last step we conclude that for sufficiently large  $n$  at least  $\frac{1}{2} - \delta$  of all  $a \in L_n$  is in  $Z$ .

**Step 1. The set  $Y$ .**

Firstly, we will show that for any  $n \in \omega$  at least  $\frac{1}{2}$  of all  $a \in L_n$  is in  $Y$ . Set  $n \in \omega$  and consider the intervals  $I_a$  for  $a \in L_n$ . Let  $\{d_0, d_1, \dots\}$  be an increasing enumeration of the set  $D$ . Observe that  $J_{d_0}$  can intersect at most  $\frac{1}{3}$  of those intervals. Similarly,  $J_{d_0} \cup J_{d_1}$  can intersect at most  $\frac{1}{3} + \frac{1}{9}$  of those intervals. Generally, the union of all  $J_d$  with  $d \in D \cap (n + m + 1)$  can intersect at most  $\frac{1}{3} + \frac{1}{9} + \dots < \frac{1}{2}$  of the intervals  $I_a$  with  $a \in L_n$ . Each  $J_d$  with  $d \in D$  and  $d \geq n + m + 1$  is of length at most  $\varepsilon^{n+m+2}$ , which is equal to the length of any  $K_j^{n+1}$  for  $3 \cdot 3^{n+1} \leq j < 4 \cdot 3^{n+1}$ . Therefore, each such  $J_d$  cannot intersect more than one  $I_a$  with  $a \in L_n$ . Hence, at least  $\frac{1}{2}$  of all  $a \in L_n$  is in  $Y$ .

**Step 2. The set  $A'$ .**

In this step we show that for sufficiently large  $n \in \omega$  at least  $1 - \delta$  of all  $a \in L_n$  is in  $A'$ .

Since  $\sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^i = 1$ , there is  $k \in \omega$  such that  $\left(\sum_{i=0}^k \frac{1}{3} \left(\frac{2}{3}\right)^i\right)^{s+1} > 1 - \delta$ .

Without loss of generality, we may assume that each  $r_i$  is in  $(0, \frac{1}{7^m})$  (if some  $r_i$  is greater than  $\frac{1}{7^m}$ , then trivially each  $I_a$  is disjoint with the union of  $(r_i + I_{a'})_{a' \in A}$ ). For each  $i \leq s$  let  $(r_{i,j})_{j \in \omega} \in 7^\omega$  be the unique sequence satisfying  $r = \frac{r_{i,0}}{7^{m+1}} + \frac{r_{i,1}}{7^{m+2}} + \dots$ . For each  $i \leq s$  let also  $(q(i,j))_{j \in \omega} \subset \omega$  be the unique sequence with the following properties:

- $q(i, 0)$  is minimal with  $r_{i,q(i,0)} \neq 0$ ;
- if  $j \in \omega$  is such that  $r_{i,q(i,j)}$  is odd, then  $q(i, j+1) > q(i, j)$  is minimal with  $r_{i,q(i,j+1)} \neq 0$ ;
- if  $j \in \omega$  is such that  $r_{i,q(i,j)}$  is even, then  $q(i, j+1) > q(i, j)$  is minimal with  $r_{i,q(i,j+1)} \neq 0$ .

Those sequences are infinite, since  $r_i$ 's are not in  $\mathbb{Q}$ .

Pick elements  $p(i, j) \in \omega$  for  $i \leq s$  and  $j \leq k$  such that:

- $p(0, j) = q(0, j)$  for each  $j \leq k$ ;
- $p(i, j) = q(i, l_i + j)$  for each  $0 < i \leq s$  and  $j \leq k$ , where  $l_i = \min\{l \in \omega : q(i, l) > p(i-1, k)\}$ .

Denote  $p = p(s, k)$  and let  $p'$  be greater than  $q(0, k+1)$  and all  $q(i, l_i + k+1)$  for  $0 < i \leq s$ .

In this step we will not need  $p'$ . The only reason for defining it is to assure in the third step that if  $a \in A$  has some required properties, then for all  $a' \in A$  and  $i \leq s$  we have  $(r_i + I_{a'}) \cap J_d = \emptyset$  whenever  $d \in D$  is such that  $I_a \cap J_d \neq \emptyset$ .

Set any  $n > p$ . We will show that at least  $1 - \delta$  of all  $a \in L_n$  is in  $A'$ .

We need to define an auxiliary set  $B \subset L_n$  with  $B \subset A'$ . Consider the intervals  $K_l^{p(0,0)}$  for  $l < 3^{p(0,0)}$ . Each of them is of length  $\varepsilon^{p(0,0)+1}$  and therefore is disjoint with the union of  $(r_0 + I_a)_{a \in A}$ . Define

$$B_0^0 = \left\{ a \in L_n : \exists_{l < 3^{p(0,0)}} I_a \subset K_l^{p(0,0)} \right\}.$$

Set now any  $i \leq s$  and  $j \leq k$  with  $(i, j) \neq (0, 0)$ . There are two possible cases.

**Case 1.** If  $q(i, l_i + j - 1)$  is even, then  $p(i, j) \neq 0$  and each of the intervals  $K_l^{p(i,j)}$  for  $l < 3^{p(i,j)}$  is disjoint with the union of  $(r_i + I_a)_{a \in A}$  (note that the distance between such  $K_l^{p(i,j)}$  and any  $x \in \bigcup_{a \in A} (r_i + I + a)$  must be greater than  $\frac{1}{7^{p'+m}}$ ). Define

$$B_j^i = \left\{ a \in L_n : \exists_{l < 3^{p(i,j)}} I_a \subset K_l^{p(i,j)} \right\}.$$

**Case 2.** If  $q(i, l_i + j - 1)$  is odd, then  $p(i, j) \neq 6$  and each of the intervals  $K_l^{p(i,j)}$  for  $3^{p(i,j)} \leq l < 2 \cdot 3^{p(i,j)}$  is disjoint with the union of  $(r_i + I_a)_{a \in A}$  (note that the distance between such  $K_l^{p(i,j)}$  and any  $x \in \bigcup_{a \in A} (r_i + I + a)$  must be greater than  $\frac{1}{7^{p'+m}}$ ). Define

$$B_j^i = \left\{ a \in L_n : \exists_{3^{p(i,j)} \leq l < 2 \cdot 3^{p(i,j)}} I_a \subset K_l^{p(i,j)} \right\}.$$

Let  $B_i = B_0^i \cup \dots \cup B_k^i$  and  $B = \bigcap_{j \leq s} B_j$ . Then  $B \subset A'$ .

We want to estimate how many of all  $a \in L_n$  is in  $B$ . Denote  $\alpha = \sum_{i=0}^k \frac{1}{3} \left(\frac{2}{3}\right)^i$ .

Firstly, observe that each  $B_j^i$  contains exactly  $\frac{1}{3}$  of all  $a \in L_n$ . What is more,  $B_0^i \cup B_1^i$  contains exactly  $\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}$  of them and, generally,  $B_i$  contains exactly  $\alpha$  of all  $a \in L_n$ . Consider now  $B_0 \cap B_1$ . Similarly as above,  $B_0 \cap B_0^1$  contains exactly  $\frac{1}{3}$  of all  $a \in B_0$ ,  $B_0 \cap (B_0^1 \cup B_1^1)$  contains exactly  $\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3}$  of them and, generally,  $B_0 \cap B_1$  contains exactly  $\alpha$  of all  $a \in B_0$ .

Likewise, we show that for any  $i \leq s$  in the set  $\bigcap_{j < i} B_j \cap B_i$  there is  $\alpha$  of all  $a \in \bigcap_{j < i} B_j$ . Therefore,  $(\alpha)^s > 1 - \delta$  of all  $a \in L_n$  is in  $B \subset A'$ .

**Step 3. The set  $Z$ .**

By the last two steps we know that at least  $\frac{1}{2} - \delta$  of all  $a \in L_n$  is in  $Y \cap A'$  whenever  $n > p$ . Observe that the set

$$F = \{a \in Y : \exists_{d \in D \cap (p'+m)} I_a \cap J_d \neq \emptyset\}$$

is finite (actually, of cardinality at most  $p'$ , by the definition of  $Y$  and the fact that  $D \subset \omega \setminus m$ ) and let  $N$  be greater than  $p$  and  $\max\{n \in \omega : \exists a \in F a \in L_n\}$ . Pick any  $n > N$  and let  $B \subset L_n$  be as in the second step. Now we only need to observe that  $Y \cap B \subset Z$ , i.e., for each  $a \in L_n$  with  $a \in Y \cap B$  we have

$$\forall i \leq s \forall_{\substack{a', a'' \in A \\ a' \neq a''}} (r_i + I_{a'}) \cap J_d = \emptyset = I_{a''} \cap J_d$$

whenever  $d \in D$  is such that  $I_a \cap J_d \neq \emptyset$ . This finishes the entire proof.  $\square$

**Corollary 2.3.** *Suppose that  $I$  is an interval of length  $\frac{1}{\gamma^m}$  for some  $m \in \omega$  and  $A \subset \omega$  is of positive density with  $\min A > m$ . Let the sequence of intervals  $(I_a)_{a \in A}$  be defined according to Spacing Algorithm. Then for any  $D \subset \omega \setminus m$  and a sequence of intervals  $(J_d)_{d \in D}$  with  $|J_d| \leq \frac{1}{\gamma^{d+1}}$  for all  $d \in D$ , if  $\underline{d}(D) < \frac{d(A)}{4}$ , then there is  $a \in A$  such that  $I_a \cap \bigcup_{d \in D \cap a} J_d$  is empty.*

*Proof.* Denote  $Y = Y((I_a)_{a \in A}, (J_d)_{d \in D})$  and define

$$\delta = \frac{d(A)}{4} - \frac{1}{2} \left( \frac{d(A)}{4} - \underline{d}(D) \right).$$

Firstly, observe that by the Spacing Algorithm  $\underline{d}(Y) \geq \frac{d(A)}{4}$  (since  $Y$  contains a subset of lower asymptotic density at least  $\frac{d(A)}{4}$ ). Therefore, there is  $n_0 \in \omega$  such that for every  $j > n_0$  we have  $\frac{\text{card}(Y \cap (j+1))}{j+1} > \delta$ . On the other hand,  $\underline{d}(D) < \frac{d(A)}{4}$ , and hence there is  $n_1 \in \omega$  such that for every  $i > n_1$  one can find  $j > i$  with  $\frac{\text{card}(D \cap (j+1))}{j+1} < \delta$ .

Put  $n = \max\{n_0, n_1\}$  and pick any  $i > n$ . Then there is  $j > i$  such that  $\frac{\text{card}(D \cap (j+1))}{j+1} < \delta$  but  $\frac{\text{card}(Y \cap (j+1))}{j+1} > \delta$ . Hence,  $\text{card}(D \cap (j+1)) < \text{card}(Y \cap (j+1))$ . By the definition of the set  $Y$ , each  $J_d$  with  $d \in D \cap (j+1)$  can intersect at most one  $I_a$  with  $a \in Y \cap (j+1)$ , so there must be some  $a \in Y \cap (j+1)$  such that  $I_a \cap \bigcup_{d \in D \cap (j+1)} J_d$  is empty. Then also  $I_a \cap \bigcup_{d \in D \cap a} J_d$ . This finishes the proof.  $\square$

### 3. ADDITIVITY OF THE IDEAL OF MICROSCOPIC SETS

In this section we proceed to our main results.

**Theorem 3.1.** *There is a bounded microscopic set which does not admit a cover of lower asymptotic density zero.*

*Proof.* Let  $I = [0, 1]$  and  $\varepsilon = \frac{1}{7}$ . The construction of the required set  $X$  is as follows. We inductively define intervals  $I_j^n$  for  $n \in \omega$  and  $j \in 2^n \cdot (\omega + 1)$ . At the end, we will put  $X = \bigcap_{i \in \omega} \bigcup_{j \in 2^i \cdot (\omega + 1)} I_j^i$ .

At the first step, apply the Spacing Algorithm for  $I$  and  $(\omega + 1)$  (note that  $|I_{-1}^{-1}| = \varepsilon^0$  and  $0 < \min(\omega + 1)$ ) to get closed intervals  $I_j^0$  for  $j \in (\omega + 1)$  with  $|I_j^0| = \varepsilon^j$ . In the  $n$ -th step of the induction (for  $n > 0$ ) we construct



a partition  $(A_j^{n-1})_{j \in \omega}$  of the set  $2^n \cdot (\omega + 1)$  and a sequence of intervals  $(I_j^n)_{j \in 2^n \cdot (\omega + 1)}$  such that  $|I_j^n| = \varepsilon^j$ . The relation between elements of the partitions and the family of intervals is as follows:

$$I_j^n \subset I_{2^{n-1}(k+1)}^{n-1} \Leftrightarrow j \in A_k^{n-1}.$$

So suppose now that  $A_k^i$  and  $I_j^{i+1}$  are defined for all  $i < m$ ,  $k \in \omega$  and  $j \in 2^{i+1} \cdot (\omega + 1)$ . Let  $A_j^m = 2^{m+j+1} + 2^{m+j+2} \cdot \omega$  for all  $j \in \omega$ . Then  $(A_j^m)_{j \in \omega}$  is a partition of  $2^{m+1} \cdot (\omega + 1)$  into sets of positive density. For each  $n \in \omega$  apply the Spacing Algorithm for  $I_{2^m(n+1)}^m$  and  $A_n^m$  (note that  $2^m(n+1) < 2^{m+n+1} = \min A_n^m$ ) to get closed intervals  $I_j^{m+1}$  for  $j \in 2^{m+1} \cdot (\omega + 1)$  with  $|I_j^{m+1}| = \varepsilon^j$  and

$$I_j^{m+1} \subset I_{2^m(n+1)}^m \Leftrightarrow j \in A_n^m.$$

Finally, define the sets

$$X_i = \bigcup_{j \in 2^i \cdot (\omega + 1)} I_j^i \text{ and } X = \bigcap_{i \in \omega} X_i.$$

Then  $X$  is a bounded microscopic set. Indeed, given  $\varepsilon' > 0$  one can find  $m > 1$  with  $\varepsilon^{2^m} < \varepsilon'$ . Then it suffices to note that the sequence of intervals  $(I_{2^m(j+1)}^m)_{j \in \omega}$  covers  $X_m$  (and hence the whole set  $X$ ) and

$$|I_{2^m(j+1)}^m| = \varepsilon^{2^m(j+1)} < (\varepsilon')^{(j+1)}.$$

Now let  $D \subset \omega$  be of lower asymptotic density zero and  $(J_d)_{d \in D}$  be a sequence of intervals such that  $|J_d| \leq \varepsilon^{d+1}$  for all  $d \in D$ . We will show that  $(J_d)_{d \in D}$  cannot cover the set  $X$  by constructing an increasing sequence  $(j_n)_{n \in \omega} \subset \omega$  with  $I_{j_n}^n \supset I_{j_{n+1}}^{n+1}$  and  $I_{j_n}^n \cap \bigcup_{d \in D \cap j_n} J_d = \emptyset$ . Then  $\bigcap_{n \in \omega} I_{j_n}^n$  will define a point from  $X$  which does not belong to  $\bigcup_{d \in D} J_d$ .

The construction of the sequence  $(j_n)_{n \in \omega}$  is as follows. By Corollary 2.3 (applied to  $I$ ,  $(\omega + 1)$  and the sequence of intervals  $(J_d)_{d \in D}$ ), there is  $j_0 \in (\omega + 1)$  such that  $I_{j_0}^0 \cap \bigcup_{d \in D \cap j_0} J_d = \emptyset$ . Assume now that  $j_i$  for  $i \leq n$  are as needed. Again, by Corollary 2.3 (applied to  $I_{j_n}^n$ ,  $A_{j_n/2^n-1}^n$  and the sequence of intervals  $(J_d)_{d \in D \setminus j_n}$ ), we can find  $j_{n+1} \in A_{j_n/2^n-1}^n$  (hence  $I_{j_n}^n \supset I_{j_{n+1}}^{n+1}$ ) with  $I_{j_{n+1}}^{n+1} \cap \bigcup_{d \in D \cap (j_{n+1} \setminus j_n)} J_d = \emptyset$ . Then also  $I_{j_{n+1}}^{n+1} \cap \bigcup_{d \in D \cap j_{n+1}} J_d = \emptyset$ , by  $I_{j_n}^n \supset I_{j_{n+1}}^{n+1}$  and the induction assumption. This ends the construction and the entire proof.  $\square$

We are ready to prove the main theorem of this paper.

**Theorem 3.2.** *Additivity of the ideal of microscopic sets is equal to  $\omega_1$ .*

*Proof.* Recall that  $\text{add}(\mathcal{M}) \geq \omega_1$  (cf. Remark 1.3 and the discussion below it). Therefore, it suffices to prove that there is a family of cardinality  $\omega_1$  consisting of microscopic sets and such that its union is not microscopic.

Define the set

$$T = \{(i, j) : i \in \omega \text{ and } j \in 2^i \cdot (\omega + 1)\} \cup \{(-1, -1)\}$$

and put  $I_{-1}^{-1} = [0, 1]$ . Let  $X$  and  $I_j^i$  for  $(i, j) \in T$  be as in the proof of Theorem 3.1 and pick a family  $\{r_\alpha : \alpha < \omega_1\} \subset (0, 1)$  such that  $r_\alpha - r_\beta \notin \mathbb{Q}$  whenever  $\alpha \neq \beta$ . Define  $X_\alpha = r_\alpha + X$  for all  $\alpha < \omega_1$ . Clearly, each  $X_\alpha$  is microscopic. We will show that  $\bigcup_{\alpha < \omega_1} X_\alpha$  is not microscopic.

Set  $\varepsilon = \frac{1}{7}$  and any sequence of intervals  $(J_n)_{n \in \omega}$  such that  $|J_n| \leq \varepsilon^{n+1}$  for all  $n \in \omega$ . Assume that  $(J_n)_{n \in \omega}$  covers  $\bigcup_{\alpha < \omega_1} X_\alpha$ .

Consider the case that there is  $\alpha < \omega_1$  such that for any pair  $(n, m) \in T$  if  $(r_\alpha + I_m^n) \cap \bigcup_{k < m} J_k = \emptyset$ , then one can find  $l \in 2^{n+1} \cdot (\omega + 1)$  such that  $(r_\alpha + I_l^{n+1}) \subset (r_\alpha + I_m^n)$  and  $(r_\alpha + I_l^{n+1}) \cap \bigcup_{k < l} J_k = \emptyset$ . This condition allows us to construct an increasing sequence  $(m_n)_{n \in \omega}$  such that  $(r_\alpha + I_{m_{n+1}}^{n+1}) \subset (r_\alpha + I_{m_n}^n)$  and  $(r_\alpha + I_{m_n}^n) \cap \bigcup_{k < m_n} J_k = \emptyset$  for all  $n \in \omega$ . Hence, the intersection  $\bigcap_{n \in \omega} (r_\alpha + I_{m_n}^n)$  defines a point from  $X_\alpha$  (and hence from  $\bigcup_{\alpha < \omega_1} X_\alpha$ ) which is disjoint with the union  $\bigcup_{k \in \omega} J_k$ .

Therefore, we can assume from now on that for any  $\alpha < \omega_1$  there is a pair  $(n_\alpha, m_\alpha) \in T$  such that  $(r_\alpha + I_{m_\alpha}^{n_\alpha}) \cap \bigcup_{k < m_\alpha} J_k = \emptyset$  but  $(r_\alpha + I_j^{n_\alpha+1}) \cap \bigcup_{k < j} J_k \neq \emptyset$  whenever  $j \in 2^{n_\alpha+1} \cdot (\omega + 1)$  is such that  $(r_\alpha + I_j^{n_\alpha+1}) \subset (r_\alpha + I_{m_\alpha}^{n_\alpha})$  (note that trivially  $(r_\alpha + I_{-1}^{-1}) \cap \bigcup_{k < -1} J_k = \emptyset$  for all  $\alpha < \omega_1$ ). There are only countably many possible choices for  $(n_\alpha, m_\alpha)$ , so one can find an uncountable  $F \subset \omega_1$  and a pair  $(n, m) \in T$  such that  $(n, m) = (n_\alpha, m_\alpha)$  for all  $\alpha \in F$ .

Define the set  $A = \{a \in 2^{n+1} \cdot (\omega + 1) : I_a^{n+1} \subset I_m^n\}$  (note that  $A = A_{m/2^{n-1}}^m$  in the notation from the proof of Theorem 3.1). By the construction of the set  $X$  we have  $d(A) > 0$ . Let  $s \in (\omega + 1)$  be such that  $\frac{1}{s} \leq \frac{d(A)}{4}$  and pick  $\alpha_0, \dots, \alpha_s \in F$  with  $r_{\alpha_0} < r_{\alpha_1} < \dots < r_{\alpha_s}$ . For each  $i \leq s$  let  $Y_i \subset A$  be the set  $Y((r_{\alpha_i} + I_a^{n+1})_{a \in A}, (J_k)_{k \in \omega})$  (cf. Definition 2.1). Let also  $Z_i$ , for  $i \leq s$ , be the set of those  $a \in Y_i$  which have the property that given any  $k \in \omega$  if  $(r_{\alpha_i} + I_a^{n+1}) \cap J_k \neq \emptyset$ , then there are no  $i < j \leq s$  and  $a' \in A$  such that  $(r_{\alpha_j} + I_{a'}^{n+1}) \cap J_k \neq \emptyset$  (hence  $Z_s = Y_s$ ).

By the Spacing Algorithm, for each  $i \leq s$  the set  $Z_i$  has lower asymptotic density at least  $\frac{d(A)}{4} \geq \frac{1}{s}$ . Define

$$Z'_i = \{k \in \omega : \exists a \in Z_i (r_{\alpha_i} + I_a^{n+1}) \cap J_k \neq \emptyset\}$$

for all  $i \leq s$ . Those sets also have lower asymptotic density at least  $\frac{1}{s}$ . Indeed, set any  $i \leq s$  and consider a bijection  $\phi$  between  $Z_i$  and  $Z'_i$  such that  $\phi(a)$  is equal to  $k \in Z'_i$  if  $(r_{\alpha_i} + I_a^{n+1}) \cap J_k \neq \emptyset$  for  $a \in Z_i$ . This function is well defined, since  $k$  with the above property is unique by the definition of  $Y_i$ . Observe that  $(r_{\alpha_i} + I_a^{n+1}) \cap \bigcup_{k < a} J_k \neq \emptyset$  for all  $a \in A$  by  $\alpha_i \in F$  and

the definition of  $(n, m)$ . Therefore  $\phi(a) \leq a$  for all  $a \in Z_i$ . It follows that  $\underline{d}(Z'_i) \geq \underline{d}(Z_i) \geq \frac{1}{s}$ .

Moreover,  $Z'_i \cap Z'_j = \emptyset$  for  $i < j \leq s$ . Indeed, if  $k \in Z'_i$ , then  $(r_{\alpha_i} + I_a^{n+1}) \cap J_k \neq \emptyset$  for some  $a \in Z_i$ , and hence (by the definition of  $Z_i$ ) there is no  $a' \in A$  such that  $(r_{\alpha_j} + I_{a'}^{n+1}) \cap J_k \neq \emptyset$ , which means that  $k \notin Z'_j$ .

Therefore,  $\{Z'_0, \dots, Z'_s\}$  is a family of  $s+1$  pairwise disjoint subsets of  $\omega$ , each of which is of lower asymptotic density at least  $\frac{1}{s}$ . A contradiction. Hence,  $(J_n)_{n \in \omega}$  cannot cover the set  $\bigcup_{\alpha < \omega_1} X_\alpha$ .  $\square$

#### 4. SOME GENERALIZATIONS OF THE IDEAL OF MICROSCOPIC SETS

In this section we investigate additivity of two ideals closely related to  $\mathcal{M}$ .

**Definition 4.1.** A set  $M \subset \mathbb{R}$  is in  $\mathcal{M}_{\text{ln}}$  if for each  $\varepsilon > 0$  there exists a sequence of intervals  $(J_n)_{n \in \omega}$  covering  $M$  and such that  $|J_n| \leq \varepsilon^{\ln(n+2)}$  for each  $n \in \omega$ .

**Definition 4.2.** A set  $M \subset \mathbb{R}$  is in  $\mathcal{M}'$  if for each  $\varepsilon > 0$  there exists  $D \subset \omega$  of asymptotic density zero and a sequence of intervals  $(J_n)_{n \in D}$  such that  $M \subset \bigcup_{n \in D} J_n$  and  $|J_n| \leq \varepsilon^{n+1}$  for each  $n \in D$ .

Recently, Horbachewska in [7] defined the so-called  $(f_n)_{n \in \omega}$ -microscopic sets. This concept was deeply studied in [5]. Let us point out that in the terminology of [7],  $\mathcal{M}_{\text{ln}}$  is the family of all  $(x^{\ln(n+2)})_{n \in \omega}$ -microscopic sets.

Observe that  $\mathcal{S} \subset \mathcal{M}' \subset \mathcal{M} \subset \mathcal{M}_{\text{ln}}$ . In fact, all inclusions are proper. One can easily construct a compact microscopic set of cardinality  $2^\omega$ , which shows that  $\mathcal{S} \neq \mathcal{M}'$ . Theorem 3.1 gives us an example of a microscopic set not belonging to  $\mathcal{M}'$ . Finally,  $\mathcal{M} \neq \mathcal{M}_{\text{ln}}$  will follow from the fact that  $\mathcal{M}_{\text{ln}}$  has additivity  $2^\omega$  under Martin's axiom (cf. Proposition 4.5).

The following lemma will be useful in our further considerations.

**Lemma 4.3.** Set  $M \in \mathcal{M}_{\text{ln}}$  and  $\varepsilon \in (0, 1)$ . Suppose that  $(J_d)_{d \in D}$  is such that  $d(D) = 0$  and  $|J_d| \leq \varepsilon^{\ln(d+2)}$  for all  $d \in D$ . Then there are  $E \subset \omega$  disjoint with  $D$  and of asymptotic density zero and a sequence of intervals  $(J_e)_{e \in E}$  covering  $M$  and such that  $|J_e| \leq \varepsilon^{\ln(e+2)}$  for each  $e \in E$ .

*Proof.* Take any  $(J_d)_{d \in D}$  such that  $d(D) = 0$  and  $|J_d| \leq \varepsilon^{\ln(d+2)}$  for all  $d \in D$ . Since  $d(D) = 0$ , there is  $k \in \omega$  such that  $\frac{\text{card}(D \cap (j+1))}{j+1} \leq \frac{1}{4}$  for all  $j > k$ . Find  $m \in \omega$  such that  $2^m > k$  and  $m \geq 2$ . We inductively pick a sequence  $(t_i)_{i \in \omega}$  of pairwise distinct elements of  $\omega \setminus D$  satisfying  $\frac{1}{2}(i+2)^m \leq t_i + 2 \leq (i+2)^m$ .

The construction is as follows. Let  $t_0 \in \omega \setminus D$  be maximal such that  $t_0 + 2 \leq 2^m$ . Note that at most one in four of all  $n \leq 2^m$  is in  $D$ , hence

$t_0 + 2 \geq \frac{1}{2}2^m$ . Assume now that  $t_0, \dots, t_{i-1}$  are constructed. Pick  $t_i \in \omega \setminus (D \cup \{t_0, \dots, t_{i-1}\})$  to be maximal such that  $t_i + 2 \leq (i + 2)^m$ . Note that at most one in four of all  $n \leq (i + 2)^m$  is in  $D$ . Moreover, until this moment we have picked only  $i$  numbers from  $\omega \setminus D$  and  $\frac{i}{(i+2)^m} < \frac{1}{4}$ , so less than one in four of all  $n \leq (i + 2)^m$  is one of the  $t_j$ 's for  $j < i$ . Therefore,  $\frac{1}{2}(i + 2)^m \leq t_i + 2$ .

Define  $E = \{t_i : i \in \omega\}$ . Obviously,  $D \cap E = \emptyset$ . What is more,  $d(E) = 0$ . Indeed, given any  $j \in \omega$ , the number of elements of the set  $E \cap (j + 1)$  is bounded above by  $(2(j + 2))^{\frac{1}{m}} - 1$ , since  $j < \frac{1}{2}(i + 2)^m - 2 \leq t_i$  whenever  $i > (2(j + 2))^{\frac{1}{m}} - 2$ . Now it suffices to observe that:

$$\frac{\text{card}(E \cap (j + 1))}{j + 1} \leq \frac{(2(j + 2))^{\frac{1}{m}} - 1}{j + 1} \rightarrow 0.$$

Since the set  $M$  is in  $\mathcal{M}_{\text{ln}}$ , there is a sequence of intervals  $(I_n)_{n \in \omega}$  covering  $M$  and such that  $|I_n| \leq (\varepsilon^m)^{\ln(n+2)} = \varepsilon^{\ln(n+2)^m}$ . Let  $J_{t_n} = I_n$  for all  $n \in \omega$  and note that for all  $n \in \omega$  we have  $|I_n| \leq \varepsilon^{\ln(t_n+2)}$ , since  $t_n + 2 \leq (n + 2)^m$  and  $\varepsilon \in (0, 1)$ .  $\square$

**Proposition 4.4.** *Both  $\mathcal{M}_{\text{ln}}$  and  $\mathcal{M}'$  are  $\sigma$ -ideals.*

*Proof.* Firstly, assume that  $(M_k)_{k \in \omega} \subset \mathcal{M}'$  and set any  $\varepsilon > 0$ . Similarly as in Remark 1.3, for each  $k \in \omega$  we can find a sequence of intervals  $(J_d)_{d \in D_k}$ , where  $D_k \subset 2^{k+1} \cdot (\omega + 1)$ , which covers  $M_k$  and satisfies  $|J_d| \leq \varepsilon^{d+1}$  for each  $d \in D_k$ . Let  $D'_k = D_k - 2^k$  and note that  $(D'_k)_{k \in \omega}$  is a family of pairwise disjoint subsets of  $\omega$ . Let also  $J'_d = J_{d+2^k}$  whenever  $d \in D'_k$  and define  $D = \bigcup_{k \in \omega} D'_k$ . Then  $(J'_n)_{n \in D}$  covers  $\bigcup_{k \in \omega} M_k$  and  $|J'_n| \leq \varepsilon^{n+1}$  for each  $n \in \omega$ .

We need to show that  $D$  is of asymptotic density zero. Set any  $\delta > 0$ . There is  $m \in \omega$  such that  $D^0 = \bigcup_{k > m} (2^{k+1} \cdot (\omega + 1) - 2^k)$  has asymptotic density less than  $\frac{\delta}{3}$ . Hence, there is  $j_0 \in \omega$  such that for all  $j > j_0$  we have  $\frac{\text{card}(D^0 \cap (j+1))}{j+1} < \frac{\delta}{2}$ . Denote  $D^1 = \bigcup_{k \leq m} D'_k$  and note that this set is of asymptotic density zero. Hence, there also is  $j_1 \in \omega$  such that for all  $j > j_0$  we have  $\frac{\text{card}(D^1 \cap (j+1))}{j+1} < \frac{\delta}{2}$ . Since  $D \subset D^0 \cup D^1$ , we have:

$$\frac{\text{card}(D \cap (j + 1))}{j + 1} \leq \frac{\text{card}(D^0 \cup D^1 \cap (j + 1))}{j + 1} < \delta$$

whenever  $j > \max\{j_0, j_1\}$ .

Assume now that  $(M_k)_{k \in \omega} \subset \mathcal{M}_{\text{ln}}$  and set any  $\varepsilon > 0$ . By Lemma 4.3 (applied to  $D = \emptyset$ ), there are  $E_0 \subset \omega$  of asymptotic density zero and a sequence of intervals  $(J_e)_{e \in E_0}$  covering  $M_0$  and such that  $|J_e| \leq \varepsilon^{\ln(e+2)}$  for each  $e \in E_0$ . However, by Lemma 4.3 there also are  $E_1 \subset \omega$  disjoint with

$E_0$  of asymptotic density zero and a sequence of intervals  $(J_e)_{e \in E_1}$  covering  $M_1$  and such that  $|J_e| \leq \varepsilon^{\ln(e+2)}$  for each  $e \in E_1$ . In this way we inductively construct a family  $(E_n)_{n \in \omega}$  of pairwise disjoint subsets of  $\omega$  and a sequence of intervals  $(J_e)_{e \in E}$ , where  $E = \bigcup_{n \in \omega} E_n$ , covering  $\bigcup_{n \in \omega} M_n$  and such that  $|J_e| \leq \varepsilon^{\ln(e+2)}$  for each  $e \in E$ .  $\square$

Now we will calculate additivity of the ideals  $\mathcal{M}_{\text{ln}}$  and  $\mathcal{M}'$  under Martin's axiom.

**Proposition 4.5.** *Assume  $\text{MA}_\kappa$ . If  $\mathcal{F} \subset \mathcal{M}_{\text{ln}}$  is a family of cardinality  $\kappa$ , then  $\bigcup \mathcal{F} \in \mathcal{M}_{\text{ln}}$ . Therefore,  $\text{add}(\mathcal{M}_{\text{ln}}) = 2^\omega$  under Martin's axiom.*

The proof is an adaptation of the proof of [4, Theorem 2.1]. Therefore, we omit some details and focus only on the modified parts.

*Proof.* Let  $\mathcal{F} = \{M_\alpha : \alpha < \kappa\}$  and  $\mathcal{B}$  be the family of all open intervals with rational endpoints. Notice that  $\mathcal{B}$  is countable. Denote  $M = \bigcup_{\alpha < \kappa} M_\alpha$  and take any  $\varepsilon \in (0, 1)$ . Let

$$\mathcal{P} = \{(J_d)_{d \in D} : d(D) = 0 \text{ and } \forall d \in D (J_d \in \mathcal{B} \text{ and } |J_d| \leq \varepsilon^{\ln(d+2)})\}$$

and define the relation  $\prec$  on  $\mathcal{P}$  by:

$$(J_d)_{d \in D} \prec (J'_d)_{d \in D'} \Leftrightarrow \bigcup_{d \in D} J_d \supset \bigcup_{d \in D'} J'_d.$$

Then  $(\mathcal{P}, \prec)$  is a partial order which is c.c.c. (for details see [4]). For all  $\alpha < \kappa$  define also

$$\mathcal{D}_\alpha = \left\{ (J_n)_{n \in D} \in \mathcal{P} : M_\alpha \subset \bigcup_{n \in D} J_n \right\}.$$

We want to prove that these sets are dense.

Take any  $\alpha < \kappa$ . We will show that  $\mathcal{D}_\alpha$  is dense. Suppose that  $(J_d)_{d \in D} \in \mathcal{P}$ . By Lemma 4.3, there is  $E \subset \omega$  disjoint with  $D$  and of asymptotic density zero and a sequence of intervals  $(J_e)_{e \in E}$  covering  $M$  and such that  $|J_e| \leq \varepsilon^{\ln(e+2)}$  for each  $e \in E$ . Observe that the sequence  $(J_n)_{n \in D \cup E}$  is in  $\mathcal{D}_\alpha$  and  $(J_n)_{n \in D \cup E} \prec (J_n)_{n \in D}$ . Therefore, the set  $\mathcal{D}_\alpha$  is dense.

By  $\text{MA}_\kappa$ , there is a filter  $\mathcal{G}$  in  $\mathcal{P}$  intersecting each  $\mathcal{D}_\alpha$  for  $\alpha < \kappa$ . Let also  $I_0, I_1, \dots$  list all the intervals  $J$  such that there are  $(J_d)_{d \in D} \in \mathcal{G}$  and  $d \in D$  with  $J = J_d$  (note here that each  $J_d$  is in  $\mathcal{B}$  and recall that  $\mathcal{B}$  is countable). Then the union  $\bigcup_{n \in \omega} I_n$  covers the set  $M$ , since each  $M_\alpha$  is contained in some  $\bigcup_{d \in D} J_d$  with  $(J_d)_{d \in D} \in \mathcal{G}$ . Moreover,  $|I_n| \leq \varepsilon^{\ln(n+2)}$  for all  $n \in \omega$  (for details see [4]). Hence, the set  $M$  is in  $\mathcal{M}_{\text{ln}}$ .  $\square$

**Proposition 4.6.** *Assume  $\text{MA}_\kappa$ . If  $\mathcal{F} \subset \mathcal{M}'$  is a family of cardinality  $\kappa$ , then  $\bigcup \mathcal{F} \in \mathcal{M}'$ . Therefore,  $\text{add}(\mathcal{M}') = 2^\omega$  under Martin's axiom.*

This proof also is based on the proof of [4, Theorem 2.1] and is very similar to the proof of Proposition 4.5.

*Proof.* Let  $\mathcal{F} = \{M_\alpha : \alpha < \kappa\}$  and denote  $M = \bigcup_{\alpha < \kappa} M_\alpha$ . Similarly as in the proof of Proposition 4.5, let  $\mathcal{B}$  be the family of all open intervals with rational endpoints and take any  $\varepsilon \in (0, 1)$ . Let

$$\mathcal{P} = \left\{ (J_d)_{d \in D} : d(D) = 0 \text{ and } \forall d \in D \left( J_d \in \mathcal{B} \text{ and } |J_d| \leq \varepsilon^{d+1} \right) \right\}$$

and define the relation  $\prec$  on  $\mathcal{P}$  by:

$$(J_d)_{d \in D} \prec (J'_d)_{d \in D'} \Leftrightarrow \bigcup_{d \in D} J_d \supset \bigcup_{d \in D'} J'_d.$$

Then, as in the proof of Proposition 4.5,  $(\mathcal{P}, \prec)$  is a partial order which is c.c.c. For all  $\alpha < \kappa$  define also

$$\mathcal{D}_\alpha = \left\{ (J_d)_{d \in D} \in \mathcal{P} : M_\alpha \subset \bigcup_{d \in D} J_d \right\}.$$

Now it suffices to prove that these sets are dense.

Take any  $\alpha < \kappa$ . We want to show that  $\mathcal{D}_\alpha$  is dense. Suppose that  $(J_d)_{d \in D} \in \mathcal{P}$ . Since  $d(D) = 0$ , there is  $m \in \omega$  such that  $\frac{\text{card}(D \cap (j+1))}{j+1} < \frac{1}{4}$  for all  $j \geq m$ . We can additionally assume that  $m \geq 4$  and  $m$  is even. Since the set  $M_\alpha$  is in  $\mathcal{M}'$ , there is  $E \subset \omega$  of density zero and a sequence of intervals  $(I_e)_{e \in E}$  covering  $M_\alpha$  and such that  $|I_e| \leq (\varepsilon^m)^{e+1} = \varepsilon^{m(e+1)}$  for all  $e \in E$ . Let  $\{e_0, e_1, \dots\}$  be an increasing enumeration of the set  $E$ . We inductively pick a sequence  $(t_i)_{i \in \omega}$  of pairwise distinct elements of  $\omega \setminus D$  satisfying  $\frac{1}{2}m(e_i + 1) \leq t_i + 1 \leq m(e_i + 1)$ .

The construction is as follows. Let  $t_0 \in \omega \setminus D$  be maximal such that  $t_0 + 1 \leq m(e_0 + 1)$ . Note that at most one in four of all  $n \leq m(e_0 + 1)$  is in  $D$ , and hence  $t_0 + 1 \geq \frac{1}{2}m(e_0 + 1)$ . Assume now that  $t_0, \dots, t_{i-1}$  are constructed. Pick  $t_i \in \omega \setminus (D \cup \{t_0, \dots, t_{i-1}\})$  to be maximal such that  $t_i + 1 \leq m(e_i + 1)$ . Note that at most one in four of all  $n \leq m(e_i + 1)$  is in  $D$ . Moreover, until this moment we have picked only  $i$  numbers from  $\omega \setminus D$  and by the fact that  $m \geq 4$ , we have  $\frac{i}{m(e_i + 1)} \leq \frac{e_i + 1}{m(e_i + 1)} < \frac{1}{4}$ . Hence, less than one in four of all  $n \leq m(e_i + 1)$  is one of the  $t_j$ 's for  $j < i$ . Therefore,  $\frac{1}{2}m(e_i + 1) \leq t_i + 1$ .

Define  $F = \{t_i : i \in \omega\}$ . Obviously,  $D \cap F = \emptyset$ . What is more,  $d(F) = 0$ . Indeed, given any  $j \in \omega$ , the number of elements of the set  $F \cap (j+1)$  is bounded above by the cardinality of the set  $\{i \in \omega : e_i \leq \frac{2(j+1)}{m} - 1\} = E \cap \frac{2(j+1)}{m}$ , since  $j+1 < \frac{1}{2}m(e_i + 1) \leq t_i + 1$  whenever  $e_i > \frac{2(j+1)}{m} - 1$ . Now

it suffices to observe that:

$$\frac{\text{card}(F \cap (j+1))}{j+1} \leq \frac{\text{card}(E \cap \frac{2(j+1)}{m})}{j+1} \leq \frac{\text{card}(E \cap (j+1))}{j+1} \rightarrow 0,$$

since  $E$  is of asymptotic density zero.

Let  $J_{t_i} = I_{e_i}$  for all  $i \in \omega$  and note that we have  $|J_{t_i}| \leq \varepsilon^{(t_i+1)}$ , since  $t_i + 1 \leq m(e_i + 1)$ . Then the sequence  $(J_n)_{n \in D \cup F}$  is in  $\mathcal{D}_\alpha$  and  $(J_n)_{n \in D \cup F} \prec (J_n)_{n \in D}$ . Therefore, the set  $\mathcal{D}_\alpha$  is dense.

The rest of the proof is similar to the proof of Proposition 4.5.  $\square$

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